extraordinarily high values of solar radiation, but it has been not only a cold winter but a cloudy winter. Hence it may have been that the direct effect of the outburst of solar activity was to produce excessive cloudiness which by high reflection diminished the radiation available to warm the earth.

In the preceding table I give the mean values of the solar radiation above mentioned. In each month I have indicated the successive five day periods by the capital letters A, B, C, D, E, and F. The values given are the number of thousandths of a calorie by which the solar radiation of a given time interval exceeds 1.900. Thus, for the first period of June the mean value is 1.946.

March Values.—On or about March 22, great sunspot activity was reported. On March 22 and 23 there were intense magnetic disturbances affecting all observations of terrestrial magnetism and the operation of telegraphs and cables. Remarkable auroral displays followed. In connection with these conditions it is interesting to note the very unusual progress of the solar constant of radiation during the month of March. This is given in the following table

Date	Mean, 11 to 17	18	19	20	21	22	23	24
Value	1.968	1.954	1.940	1.931	1.941	1.927	1.866	1.905

It is highly probable that the results just given will have a special significance in connection with the remarkable outbreak of solar activity to which attention has been drawn.

THE PERMANENT GRAVITATIONAL FIELD IN THE EINSTEIN THEORY

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1. In accordance with the theory of Einstein a permanent gravitational field is defined by a quadratic differential form

$$ds^2 = \sum_{i,k}^{1,..4} g_{ik}dx_i dx_k, \ (g_{ik} = g_{ki}), \tag{1}$$

where the g's, called the potentials of the field, are determined by the condition of satisfying ten partial differential equations of the second order, $G_{ik} = O$. When the four coördinates x_i are functions of a single parameter, the locus of the point with these coördinates is a curve in four-space. If these functions are of such a character that the integral

$$\int \sqrt{\Sigma g_{ik} dx_i dx_k} \tag{2}$$

is stationary along the curve, the curve is called a "world-line," or a geodesic, in the four-space.

Einstein[1] considered the case when x_1, x_2, x_3, are rectangular coördinates and x_4 represents the time, and assumed that the field was produced by a mass at the origin which did not vary with the time. In order to obtain the equations of the world-lines in the form which enabled him to establish his well-known expression for the precession of the perihelion of Mercury, Einstein made also the following assumptions:

A. The quantities g_{ik} are independent of t.

B. The equations $g_{i4} = 0$ for $i = 1, 2, 3$.

C. The solution is spacially symmetric with respect to the origin of coördinates in the sense that the solution is unaltered by an orthogonal transformation of x_1, x_2, x_3.

D. The quantities $g_{ik} = 0$ for $i \pm k$ at infinity and also $g_{44} = -g_{11} = -g_{22} = -g_{33} = 1$. Schwarzschild,[2] using the first three of these assumptions and certain others integrated the equations $G_{ik} = 0$, and obtained (1) in the form

$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2(d\theta^2 + \sin^2 \theta d\varphi^2), \tag{3}$$

where α is a constant depending on the mass at the origin. Levi-Civita[3] has given three solutions of the equations $G_{ik} = 0$, one of which includes the above, and Weyl[4] has given still another solution. Later, Kottler[5] obtained the form (3) not by the solution of the equations $G_{ik} = 0$ but as a consequence of certain postulates. It is the purpose of this paper to accomplish the same result by the following set of postulates:

I. Assumptions A and B of Einstein, in accordance with which we write (1) in the form

$$ds^2 = V^2dt^2 - ds_0{}^2, \tag{4}$$

where

$$ds_0{}^2 = \sum_{ik}^{1,2,3} a_{ik}dx_idx_k, \tag{5}$$

the functions V and a_{ik} being independent of t.

II. The function V is a solution of

$$\Delta_2 V = 0, \tag{6}$$

where $\Delta_2\theta$ is the Beltrami differential parameter formed with respect to the form (5), and is defined by

$$\Delta_2\theta = \frac{1}{\sqrt{a}} \sum_{i,k}^{1,2,3} \frac{\partial}{\partial x_i}\left(\sum_{k}^{1,2,3} a^{(ik)} \sqrt{a}\, \frac{\partial\theta}{\partial x_k}\right), \tag{7}$$

where a is the determinant of the functions a_{ik} and $a^{(ik)}$ is the co-factor of a in this determinant divided by a. This assumption is equivalent to the equation $G_{44} = 0$.

III. The surfaces V = const. form part of a triply orthogonal system in the space, S_3, of coördinates x_1, x_2, x_3.

IV. The orthogonal trajectories of V = const. in S_3 are paths of the particle, in the sense that the coördinates x_1, x_2, x_3, of a world-line determine a path in S_3 of a particle in the gravitational field for which the world-line is the representation in terms of space and time t.

V. The form (5) is euclidean to a first approximation.

2. In order to simplify the equations, we take (5) in the form

$$ds_o^2 = \sum_i^{1,2,3} a_i dx_i^2. \tag{8}$$

If we choose s for the parameter along a world-line and apply the Euler equations of condition that the integral

$$\int \sqrt{V^2dt^2 - ds_o^2}$$

be stationary along the line, we get

$$\frac{d^2x_i}{ds^2} + \sum_j \frac{\partial \log a_i}{\partial x_j}\frac{dx_i}{ds}\frac{dx_j}{ds} - \frac{1}{2a_i}\sum_j \frac{\partial a_j}{\partial x_i}\left(\frac{dx_j}{ds}\right)^2 + \frac{V}{a_i}\frac{\partial V}{\partial x_i}\left(\frac{dt}{ds}\right)^2 = 0, \tag{9}$$

$$\frac{dt}{ds} = \frac{k}{V^2}, \tag{10}$$

where k is a constant. In these equations and hereafter the symbol $\sum\limits_j$ means the sum for $j = 1, 2, 3$.

We inquire under what conditions the path of a particle in S_3 is a geodesic, that is a curve along which the integral $\int \sum\limits_i a_i dx_i^2$ is stationary. When s_o is taken for the parameter along such a geodesic, we find that

$$\frac{d^2x_i}{ds_o^2} + \sum_j \frac{\partial \log a_i}{\partial x_j}\frac{dx_i}{ds_o}\frac{dx_j}{ds_o} - \frac{1}{2a_i}\sum_j \frac{\partial a_j}{\partial x_i}\left(\frac{dx_j}{ds_o}\right)^2 = 0. \tag{11}$$

From (4) and (10) it follows that the parameters s and s_o along a world-line and the corresponding path in S_3 are in the relation

$$ds_o = \sqrt{\frac{k^2}{V^2} - 1}\, ds. \tag{12}$$

With the aid of this relation we find that in order that equations (9) and (11) define the same curves in S_3, we must have

$$\frac{dV}{ds_o}\frac{dx_i}{ds_o} - \frac{1}{a_i}\frac{\partial V}{\partial x_i} = 0 \quad (i = 1, 2, 3). \tag{13}$$

These are the conditions that the path be an orthogonal trajectory of the surfaces V = const. When we require that equations (11) and (13) be consistent, we find that it is necessary and sufficient that ΔV be a function of V, say

$$\Delta V = \varphi(V), \tag{14}$$

where ΔV is the first differential parameter of V with respect to the form ds^2_0. For the form (5) its expression is

$$\Delta V = \sum_{ik} a^{(ik)} \frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_k}. \tag{15}$$

When (14) is satisfied by a function V, the surfaces V = const. are said to be *geodesically parallel*. Hence we have the theorem:

A necessary and sufficient condition that a path in a gravitational field be a geodesic in S_3 is that it be one of the orthogonal trajectories of the surfaces V = const., which must form a geodesically parallel family.

3. The function V is interpreted as the velocity of light in the field, and consequently along a world-line of a ray of light $ds = 0$, as follows from (4). In order to find the equations of these world-lines we apply the Fermat principle that $\int dt$ be stationary along such a line, that is the integral $\int\sqrt{\frac{1}{V^2}\Sigma a_i dx_i^2}$. This gives the equations

$$\frac{d^2x_i}{dt^2} + \frac{dx_i}{dt}\sum_j \frac{\partial}{\partial x_j}\log\frac{a_i}{V^2}\frac{dx_j}{dt} - \frac{V^2}{2a_i}\sum_j\frac{\partial}{\partial x_i}\left(\frac{a_j}{V^2}\right)\left(\frac{dx_j}{dt}\right)^2 = 0.$$

When we require that the path of a ray of light be a geodesic in S_3, we obtain equations (13) and (14) as formerly. Hence:

When the orthogonal trajectories of the surfaces V = const. are paths of a particle in a gravitational field, they are also the paths of a ray of light; and conversely.

4. In accordance with III we choose the coördinates x_1, x_2, x_3, so that V is a function of x_1 alone and we have (8). Then from IV, which is equivalent to (14), written $\Delta V = \varphi(x_1)$, and II we have

$$a_1 = \frac{(V')^2}{\varphi},\quad a_2 a_3 = \frac{\psi^2}{\varphi}, \tag{16}$$

where the prime indicates differentiation with respect to x_1 and ψ is independent of x_1.

If $d\bar{s}_0{}^2 = \Sigma\bar{a}_i dx_i^2$ is the linear element of euclidean space, the functions $\bar{a}_i$ must satisfy the six equations of Lame.[6] If a_1 is to be a function of x_1 alone and the second of (16) is to be satisfied, it can be shown that the coördinates x_1 can be chosen, so that

$$d\bar{s}_0{}^2 = dx_1^2 + x_1^2(\sigma^2 dx_2^2 + \tau^2 dx_3^2), \tag{17}$$

where σ and τ are independent of x_1 and satisfy

$$\frac{\partial}{\partial x_2}\left(\frac{1}{\sigma}\frac{\partial\tau}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\tau}\frac{\partial\sigma}{\partial x_3}\right) + \sigma\tau = 0. \tag{18}$$

This is the condition that the expression in parenthesis in (17) is the linear element of the unit sphere.[7]

If we write

$$V^2 = c^2(1+ 2\varphi_1(x_1))$$

where c is the constant velocity of light, and in accordance with assumption V require that (16) be satisfied by expressions approximating the coefficients in (17), we obtain $\varphi_1 = -\frac{b}{2cx_1}, \frac{1}{a_1} = 1 + 2\varphi_1$, where b is a constant. Hence, if we take $\sigma = 1$, $\tau = \sin x_2$, which is a set of solutions of (18), we obtain the form (3). Another choice gives at once one of the forms found by Levi-Civita.[3]

[1] *Berlin Sitzungsberichte*, 1915 (831).
[2] *Ibid.*, 1916 (189).
[3] *Rendiconti dei Lincei* (Ser. 5), **27**, 1918 (350).
[4] *Ann. Physik*, **54**, 1917 (117).
[5] *Ibid.*, **56**, 1918 (401).
[6] Eisenhart's *Differential Geometry*, (449).
[7] *Ibid.*, (157).

A SIMPLIFIED METHOD FOR THE STATISTICAL INTERPRETATION OF EXPERIMENTAL DATA

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One of the fundamental postulates of the law of probability of errors is that positive and negative errors are equally frequent and that, therefore, the arithmetical mean is the most probable mean. This is true when we are dealing with *small independent errors*, but in cases of *interdependent* values of natural *frequencies* (physical, biological, agricultural), it may or may not be true, depending upon the maximum deviation from the mean. Thus we can conceive of a molecule of oxygen gas with a momentary velocity of zero or of "infinity."[1] Yet the average velocity as determined from density and pressure measurements is but a few hundred yards per second, a value not very far from zero, but infinitely different from "infinity." When the frequencies of such data are plotted on rectangular coördinate paper, we obtain what are called skew curves. Within the last twenty-five years a great deal has been written concerning skew curves and their types, and, judging from the number which have thus far appeared in print, they promise to be of infinite variety. To these various types of frequency curves have been fitted mathematical formulas and with their aid statistical constants have been obtained, many of which we have reason to believe are theoretically inapplicable and practically misleading.